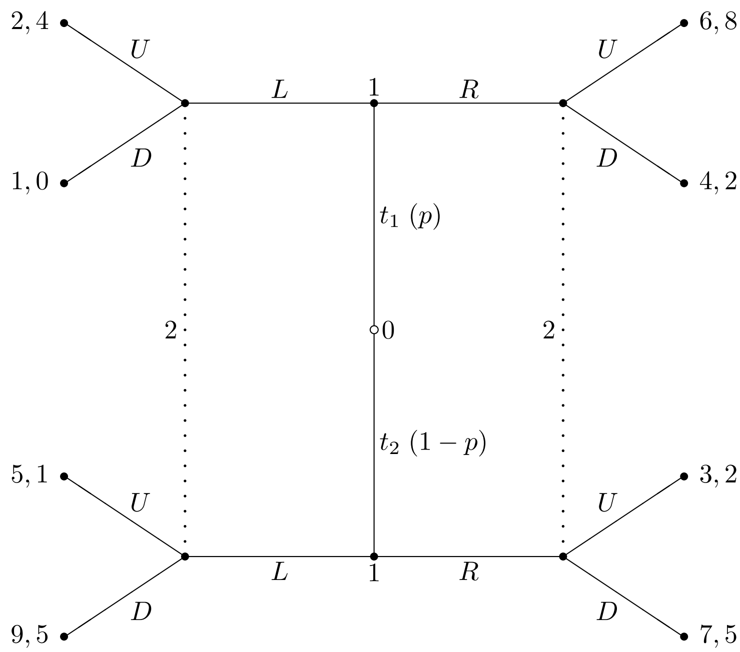
Microeconomic Theory II — Spring 2026
Final Exam SOLUTIONS — Mikhael Shor

Question 1. Consider the following game. First, nature (player 0) selects t_1 with probability p, $0 < p < 1$, or t_2 with probability $1-p$. Next, player 1 selects L or R. Lastly, player 2 selects U or D. Player 1's payoffs are listed first.



(a) Find all values of p for which a pooling weak Perfect Bayesian equilibrium exists and show one such equilibrium.

> $p \le \frac{1}{3}$. E.g., $\{t_1 \to R,\ t_2 \to R;\ R \to D,\ L \to U\},\ \mu(t_1|L) = 1,\ \mu(t_1|R) = p\}$.

First, note that for t_1, minimum utility from $R = \min\{4,6\} = 4$ is greater than the maximum utility from $L = \max\{1,2\} = 2$ so the only possible pooling equilibrium is on R.

Next, note that for t_2, R can only be optimal if $R \to D$ and $L \to U$, as otherwise L would result in higher payoffs than R.

- For $R \to D$, on-path beliefs $\mu(t_1|R) = p$. We need $EU_2(D|R) \ge EU_2(U|R) \Rightarrow 2p + 5(1-p) \ge 8p + 2(1-p) \Rightarrow p \le \frac{1}{3}$.
- For $L \to U$, we need off-path beliefs $\mu = \mu(t_1|L)$ such that $EU_2(U|L) \ge EU_2(D|L) \Rightarrow 4\mu + 1(1-\mu) \ge 0\mu + 5 - 5\mu \Rightarrow \mu(t_1|L) \ge \frac{1}{2}$.

Thus, when $p \le \frac{1}{3}$, a pooling equilibrium exists where both types play R; $R \to D$, $L \to U$; $\mu(t_1|R) = p$, $\mu(t_1|L) \ge \frac{1}{2}$.

Note 1a.1: It is important not to confuse model parameters (e.g., p) with beliefs implied by the consistency requirement (i.e., $\mu(t_1|R) = p$). A pooling equilibrium exists whenever the parameter $p \leq \frac{1}{3}$ and in such an equilibrium, $\mu(t_1|R)$ can only take on a single value, p.

Note 1a.2: Do not confuse the off-path belief constraints that sustain an equilibrium with those imposed by the intuitive criterion. A weak PBE permits any off-path belief, including ''silly'' ones. Here, $\mu(t_1|L) \geq \frac{1}{2}$ supports the equilibrium even though it may seem unreasonable. The intuitive criterion, applied in part (c), is a separate refinement that asks whether such beliefs *are* in fact silly and, if so, replaces them. Imposing that belief on the equilibrium would imply that $E_2[D|L] = 5 > 1 = E_2[U|L]$, so player 2 plays D after L, which would make a pooling equilibrium impossible.

(b) Find all values of p for which a separating weak Perfect Bayesian equilibrium exists and show one such equilibrium.

$p \in (0,1)$. $\{t_1 \to R,\ t_2 \to L;\quad R \to U,\quad L \to D\}$, $\mu(t_1|R) = \mu(t_2|L) = 1$.

As above, t_1 will always select R so the only potential separating equilibrium is $(t_1 \to R,\ t_2 \to L)$.

1. Beliefs are degenerate: $\mu(t_1|R) = \mu(t_2|L) = 1$.
2. Player 2 after R plays U $(8 > 2)$ and after L plays D $(5 > 1)$.
3. Type t_1 deviation to L: gets $(L, D) = 1 < 6$. No deviation.
   Type t_2 deviation to R: gets $(R, U) = 3 < 9$. No deviation.

Note 1b.1: The separating equilibrium logic above requires that $p \in (0,1)$. The open interval is specified by the problem $(0 < p < 1)$, since $p = 0$ or $p = 1$ would mean that there is nothing to ''separate" and that one of the information sets is off the equilibrium path.

Note 1b.2: The four-step procedure we use to derive wPBE is the method for finding an equilibrium, not itself the equilibrium statement. A weak PBE must specify: (i) a strategy for each type of player 1, (ii) a strategy for player 2 at each information set, and (iii) beliefs at every information set, even when degenerate. At the very least, a student should point to the first three steps as containing the necessary components of a wPBE.

(c) Which equilibria survive the intuitive criterion? Carefully explain.

Only the separating PBE ($t_1 \to R,\ t_2 \to L$) satisfies the intuitive criterion.

- Separating PBE: No off-equilibrium beliefs, so the intuitive criterion imposes no additional restriction. Survives IC
- Pooling: The off-path message is L.
  Type t_1 never benefits from deviating while t_2 can benefit.
  IC requires $\mu(t_1|L) = 0$ which implies Player 2 plays $L \to D$.
  But then type t_2 gets $(L, D) = 9 > 7 = (R, D)$ and deviates.
  Thus, the pooling equilibrium fails IC.

Note 1c.1: The intuitive criterion restricts off-path beliefs by asking which types *could* rationally send an unsent message. In the separating equilibrium, every message is on the equilibrium path, so there is no off-path message and the IC imposes no additional restriction. This is the complete argument. It has nothing to do with whether players ‘‘want to deviate’’ (a property already established by Nash/PBE).

Note 1c.2: The IC cannot override Bayes’ rule at on-path information sets. In the separating equilibrium, L is sent by t_2, so $\mu(t_1|L) = 0$ is already the Bayesian belief, not an IC restriction. Applying the IC as though L were unsent (as in the pooling case) confuses which equilibrium is being tested.

Note 1c.3: The intuitive criterion requires a chain of reasoning, not just a conclusion. For the pooling equilibrium, the steps are: (i) identify the off-path message (L); (ii) show that t_1 could *never* benefit from L ($\max(2, 1) = 2 < 4$), but t_2 *could* benefit ($\max(5, 9) = 9 > 7$); (iii) conclude $\mu(t_1|L) = 0$; (iv) derive player 2’s best response (D, since $5 > 1$); (v) show t_2 deviates ($9 > 7$). Skipping intermediate steps (even to a correct conclusion) does not demonstrate application of the criterion.

Question 2. Consider a principal-agent problem in which the agent chooses between two levels of effort, $e \in \{e_l, e_h\}$. The principal pays the agent a wage $w_s \geq 0$ in state s and realizes output of π_s. There are three states, with output levels $(\pi_1, \pi_2, \pi_3) = (1, 10, 49)$. The probability of a state s (or output π_s) conditional on the agent's effort is given by:

	π_1 1	π_2 10	π_3 49
e_l	$\frac{2}{9}$	$\frac{4}{9}$	$\frac{3}{9}$
e_h	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{6}{9}$

The agent's utility is $u(w, e) = \ln w - c(e)$, where $c(e_h) = \ln 4, c(e_l) = \ln 2$, and the agent's reservation utility is $\underline{u} = 0$. The principal is risk neutral with utility in state s given by $\pi_s - w_s$.

The agent's effort is unobservable.
The state is observable by both the agent and the principal.

(a) Determine the wage schedule that optimally implements e_l.

$w_s = 2$ for all s.

Risk aversion implies that the optimal contract is a fixed wage (which satisfies the low effort IC constraint) with IR binding:

$\ln w - \ln 2 = 0 \Rightarrow \ln w = \ln 2 \Rightarrow w = 2.$

Note 2a.1: Effort is unobservable, so the principal cannot condition wages on effort, only on the realized state. In the observable effort case, we couple $w^L = 2$ with a w^H sufficiently small. Here, wages are w_1, w_2, w_3, i.e., only contingent on state.

(b) Determine the wage schedule that optimally implements e_h.

$w_1 = w_2 = 1$ and $w_3 = 8$.

States 1 and 2 have the same ratios ($\frac{2/9}{1/9} = \frac{4/9}{2/9} = 2$) so $w_1 = w_2$.

IC: $(\frac{1}{9} + \frac{2}{9}) \ln w_1 + \frac{6}{9} \ln w_3 - \ln 4 \geq (\frac{2}{9} + \frac{4}{9}) \ln w_1 + \frac{3}{9} \ln w_3 - \ln 2$

$\Rightarrow \frac{1}{3}(\ln w_3 - \ln w_1) \geq \ln 4 - \ln 2 = \ln 2 \Rightarrow \ln \frac{w_3}{w_1} \geq 3 \ln 2 = \ln 8 \Rightarrow w_3 \geq 8w_1$

IR: $\frac{1}{3} \ln w_1 + \frac{2}{3} \ln w_3 - \ln 4 \geq 0$

Both bind. From IC, $w_3 = 8w_1$. Substituting into IR:
$\frac{1}{3}\ln w_1 + \frac{2}{3}\ln(8w_1) = \ln 4 \Rightarrow \frac{1}{3}\ln w_1 + \frac{2}{3}(\ln 8 + \ln w_1) = 2\ln 2$
$\Rightarrow \ln w_1 + 2\ln 2 = 2\ln 2 \;\Rightarrow\; \ln w_1 = 0$

Therefore, $w_1 = w_2 = 1$ and $w_3 = 8$. All wages are positive so the (implicit) non-negativity constraint is satisfied, though for our utility function, that constraint can never bind.

Note 2b.1: In this problem, the optimal wages in states 1 and 2 happen to be $w_1 = w_2 = 1$, which corresponds to zero utility ($\ln 1 = 0$) in those states. This is an artifact of the chosen parameters, not a general property. The correct derivation uses two binding constraints (IC and IR) to solve for two unknowns (w_1 and w_3). Assuming $w_1 = 0$ (infeasible under $\ln w$) or assuming utility equals zero in ''bad'' states substitutes a guess for a derivation. It gives the wrong answer for almost any other cost or reservation utility specification.

(c) Imagine that the government institutes a minimum wage, $\hat{w} \geq 1$, requiring that $w_s \geq \hat{w}\ \forall s$. Find the range of $\hat{w}$ for which the principal:

i. implements high effort,

ii. implements low effort, and

iii. does not contract with the agent.

i. $1 \leq \hat{w} \leq 3$:	ii. $3 < \hat{w} < 21$	iii. $\hat{w} \geq 21$

First, calculate $E[\pi]$ for each effort level:
$E[\pi|e_h] = \frac{1}{9}(1) + \frac{2}{9}(10) + \frac{6}{9}(49) = \frac{1}{9} + \frac{20}{9} + \frac{294}{9} = \frac{315}{9} = 35$.
$E[\pi|e_l] = \frac{2}{9}(1) + \frac{4}{9}(10) + \frac{3}{9}(49) = \frac{2}{9} + \frac{40}{9} + \frac{147}{9} = \frac{189}{9} = 21$.

For e_l with minimum wage, constant wage $= \max(2, \hat{w})$.
For $\hat{w} \leq 2$: $\text{Profit}_l = 19$. For $\hat{w} > 2$: $\text{Profit}_l = 21 - \hat{w}$.

For e_h with minimum wage $\hat{w} \geq 1$: The floor binds on w_1 and w_2, so $w_1 = w_2 = \hat{w}$. The binding IC gives $w_3 = 8\hat{w}$.
$\text{Profit}_h = 35 - (\frac{3}{9}\hat{w} + \frac{6}{9}(8\hat{w})) = 35 - \frac{17\hat{w}}{3}$.

High vs. low effort:
For $1 \leq \hat{w} \leq 2$: $\text{Profit}_h = 35 - \frac{17\hat{w}}{3} \geq 35 - \frac{34}{3} = \frac{71}{3} > 19 = \text{Profit}_l$.
For $\hat{w} > 2$: $35 - \frac{17\hat{w}}{3} \geq 21 - \hat{w} \;\Rightarrow\; 14 \geq \frac{14\hat{w}}{3} \;\Rightarrow\; \hat{w} \leq 3$.
so high effort is preferred to low when $\hat{w} \leq 3$

Low effort vs. no contract:
$\text{Profit}_l \leq 0 \;\Rightarrow\; 21 - \hat{w} \leq 0 \;\Rightarrow\; \hat{w} \geq 21$.

Note 2c.1: A minimum wage imposes a floor on each state's wage but it does not eliminate the need for incentive compatibility. To implement e_h, the principal must still make state 3 more rewarding than states 1 and 2. With the floor binding on w_1 and w_2, the IC relationship $w_3 = 8\hat{w}$ still determines w_3. Setting all wages equal to $\hat{w}$ provides no incentive for high effort and simply replicates the e_l contract.

Note 2c.2: The IC constraint determines a *ratio*: $w_3 = 8w_1$. When a minimum wage forces $w_1 = \hat{w}$, the high-output wage must scale accordingly: $w_3 = 8\hat{w}$. The IC is not satisfied by keeping w_3 at its original level (8) while raising w_1. That would fail to satisfy IC and thus not induce high effort.

Note 2c.3: The principal's decision is based on *profit* ($E[\pi|e]-E[w|e]$), not wages alone. High effort costs more in expected wages but also generates higher expected output (35 vs. 21). The relevant comparison is $35 - \frac{17\hat{w}}{3}$ versus $21 - \hat{w}$, not $\frac{17\hat{w}}{3}$ versus $\hat{w}$.

Note 2c.4: High effort generates more expected output ($35 > 21$) but also requires higher wages to satisfy IC and IR. The expected wage under e_h is $\frac{17\hat{w}}{3}$, which grows faster than the e_l wage of $\hat{w}$. Comparing profits while assuming identical wages across effort levels ignores the cost of incentives and results in a mere comparison of expected output which is (of course) higher under high effort. The reason a principal ever prefers e_l is precisely because it requires lower wages.

Question 3. Two identical firms (1 and 2) each currently earn steady profits from existing products. Each is considering whether to develop a next-generation product.

Stage 1. Each firm simultaneously decides whether to Innovate (I) or Not Innovate (N). Innovation requires an upfront R&D cost of 2.

Stage 2. Payoffs depend on the Stage 1 decisions:

- If **both firms innovate**, each must choose a technology platform from $\{A, B, C, D\}$ for its new product. These choices are made simultaneously, with payoffs given by:

		Firm 2 A	B	C	D
Firm 1	A	6, 6	4, 5	5, 3	7, 4
	B	5, 4	3, 3	7, 1	0, 2
	C	3, 5	1, 7	2, 2	9, 3
	D	4, 7	2, 0	3, 9	8, 8

- If **exactly one firm innovates**, the innovating firm earns 14 and the other firm earns 0.

- If **neither firm innovates**, both earn 8 from their existing products.

Each firm's total payoff equals its Stage 2 payoff minus any R&D cost.

(a) Find all subgame-perfect Nash equilibria of this two-stage game. Show each step and argue or demonstrate that no other equilibria exist.

`{ I,A ; I,A }`

First, we solve the Stage 2 subgame when both firms innovate:
D is strictly dominated by various mixtures of A, B, and C, such as $\frac{2}{5}A+\frac{3}{5}C$. By symmetry, both players' strategy D is dominated. In the reduced game, C is strictly dominated by A, and then A strictly dominates B. This game is dominance solvable. The unique Nash equilibrium of this subgame is (A, A) with payoffs $(6, 6)$. Therefore, any SPNE must prescribe (A, A) after (I, I).

Next, we consider the reduced game in Stage 1, substituting subgame outcomes and subtracting R&D costs:

	I	N
I	$6-2, 6-2$	$14-2, 0$
N	$0, 14-2$	$8, 8$

I strictly dominates N for each firm, implying that there is a unique Nash equilibrium of Stage 1.

This is a prisoner's dilemma: both firms innovate (the dominant-strategy), yet both would be better off if neither innovated ($8 > 4$).

Unique SPNE: Both firms innovate; both choose platform A. No other SPNE exists because every subgame has a unique Nash equilibrium, and Stage 1 has a strictly dominant strategy.

Note 3a.1: The question asks to ``find all subgame-perfect Nash equilibria'' and ``argue or demonstrate that no other equilibria exist.'' Finding (A, A) by checking best responses to pure strategies shows it *is* a Nash equilibrium but does not rule out others, including mixed-strategy equilibria. To establish uniqueness, one must show either that the game is dominance solvable or directly verify that no other pure or mixed equilibrium satisfies the indifference conditions. Assuming that games have only pure-strategy equilibria is a major conceptual error.

(b) What is each firm's equilibrium payoff?

Each firm earns $6 - 2 = 4$.