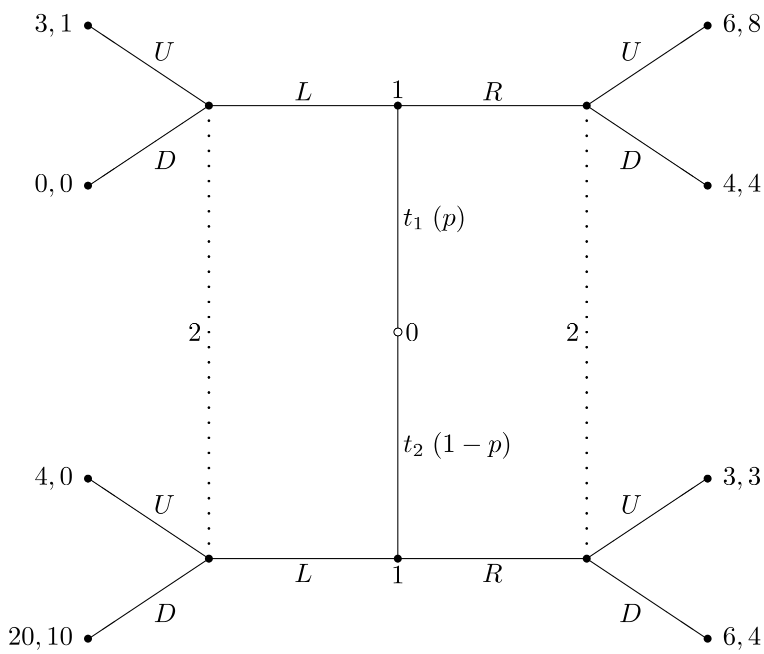
Microeconomic Theory II **Spring 2025**
Final Exam SOLUTIONS **Mikhael Shor**

Question 1. Consider the following game. First, nature (player 0) selects t_1 with probability p, $0 < p < 1$, or t_2 with probability $1-p$. Next, player 1 selects L or R. Lastly, player 2 selects U or D.



Find all values of p for which a **pooling** weak Perfect Bayesian equilibrium exists and show one such equilibrium.

$p \leq \frac{1}{5}$.
E.g., $t_1 \to R, t_2 \to R; R \to D, L \to U; \mu(t_1|L) = 1, \mu(t_1|R) = p$.

First note that t_1 will always select R since $\min(4,6) > \max(0,3)$. Therefore, the only pooling equilibrium can be on R, R. Second, for type t_2 to select R, we must have $R \to D$ (payoff of 6) and $L \to U$ (payoff of 4). Since $\mu(t_1|R) = p$, for $R \to D$ we must have $4 \geq 8p + 3(1-p)$ or $p \leq \frac{1}{5}$. For $L \to U$ we require $\mu(t_1|L) \geq \frac{10}{11}$.

Note 1.1 Be sure to distinguish between the condition on $p \ le 15$ and the belief $\mu(t_1|R) = p$ (i.e., $\mu(t_1|R) \leq \frac{1}{5}$ isn't correct as the belief on the equilibrium path is a number, not a range.

Question 2. Consider the Spence signaling model in which a worker is either a high type ($t = H$) or a low type ($t = L$), each with equal probability. The high type has productivity 4 and the low type has productivity 2. The cost of obtaining a level of education, e, is $c_H(e) = \frac{1}{8}e$ for the high type and is $c_L(e) = \frac{1}{4}e$ for the low type. Wage, w, equals a worker's expected productivity. A worker's utility function is $w - c_t(e)$.

A separating equilibrium involves each type, t, choosing a level of education e_t, where $e_L \neq e_H$. What is the range of e_H for which a **separating** weak Perfect Bayesian equilibrium exists?

$e_H \in [8, 16]$

`We require` $u_L(e_L) \geq u_L(e_H)$ `and` $u_H(e_H) \geq u_H(e_L)$ `for a separating equilibrium, and we know that` $e_L = 0$`.`

$u_L(e_L) \geq u_L(e_H) \Rightarrow 2 - 0 \geq 4 - \frac{1}{4}e_H \Rightarrow e_H \geq 8$
$u_H(e_L) \leq u_H(e_H) \Rightarrow 2 - 0 \leq 4 - \frac{1}{8}e_H \Rightarrow e_H \leq 16$

Question 3. Consider a principal-agent problem in which the agent chooses between two levels of effort, $\{e_l, e_h\}$. The principal pays the agent a wage $w_s \geq 0$ in state s and realizes output of π_s. There are four states, with output levels $(\pi_1, \pi_2, \pi_3, \pi_4) = (0, 3, 9, 18)$. The probability of a state s (or output π_s) conditional on the agent's effort is given by:

	π_1 0	π_2 3	π_3 9	π_4 18
e_l	$\frac{2}{9}$	$\frac{4}{9}$	$\frac{1}{9}$	$\frac{2}{9}$
e_h	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{4}{9}$

The agent's utility is $u(w, e) = \ln w - c(e)$, where $c(e_h) = \ln 4, c(e_l) = \ln 2$, and the agent's reservation utility is 0. The principal is risk neutral with utility in state s given by $\pi_s - w_s$.

(a) Determine the wage schedule that optimally implements e_l and the wage schedule that optimally implements e_h when effort is *observable*.

Since the agent is risk averse, the optimal wage will be constant across states and satisfy IR with equality. For low effort: $\ln w - \ln 2 = 0$ so $w_s^L = 2$ (with w_s^H sufficiently small). Similarly, for high effort, $w_s^H = 4$ (with w_s^L sufficiently small).

(b) Determine the wage schedule that optimally implements e_l when effort is *unobservable*.

$w_s = 2$. Again, wage is constant and IR constraint is binding.

Note 3.1 Note that in part (a), there are eight wages (one for each effort/state pair) and in part (b) there are four (one for each state, as effort is unobservable).

(c) Determine the wage schedule that optimally implements e_h when effort is *unobservable*.

$w_1 = w_2 = 1, w_3 = w_4 = 8$

First note that the ratios of probabilities in states 1 and 2 are the same, and in states 3 and 4 are the same. Therefore $w_1 = w_2$ and $w_3 = w_4$. The IC and IR constraints are:

IC: $\frac{1}{3}\ln w_1 + \frac{2}{3}\ln w_3 - \ln 4 \geq \frac{2}{3}\ln w_1 + \frac{1}{3}\ln w_3 - \ln 2 \Rightarrow \ln w_3 \geq \ln w_1 + \ln 8$
IR: $\frac{1}{3}\ln w_1 + \frac{2}{3}\ln w_3 - \ln 4 \geq 0$

Since both constraints must bind, substitute $\ln w_3 = \ln w_1 + \ln 8$ (or $w_3 = 8w_1$) into the IR constraint to get $\ln w_1 = \ln 4 - \frac{2}{3}\ln 8 = \ln 1$.

(d) Determine the optimal wage schedule when effort is *unobservable*.

$w_1 = w_2 = 1, w_3 = w_4 = 8$

Compare the profit from implementing low effort:
$\frac{2}{9}(0-2) + \frac{4}{9}(3-2) + \frac{1}{9}(9-2) + \frac{2}{9}(18-2) = 4\frac{1}{3}$
with the profit from implementing high effort:
$\frac{1}{9}(0-1) + \frac{2}{9}(3-1) + \frac{2}{9}(9-8) + \frac{4}{9}(18-8) = 5$

(e) Is the principal's profit higher under observable effort, higher under unobservable effort, or are the profits the same? Briefly explain intuitively.

Since implementing high effort is optimal (requiring non-constant wages) and the agent is risk averse, profit must be higher under observable effort.

Question 4. Two drivers are traveling side-by-side in two lanes on a highway that are narrowing due to a merge. As the lanes narrow, the drivers are increasingly likely to crash in the next minute unless one of them yields (slows down) and allows the other driver to pass. Each driver simultaneously selects a time $t_i \in [0, 1]$ to yield if the other driver has not yet yielded.

The probability of a crash is $\min(t_1, t_2)$. The payoff for driver i is -1 if they crash, 1 if they do not crash and driver i is in front ($t_i > t_j$) and 0 if they do not crash and driver i is behind ($t_i < t_j$), with ties resolved randomly.

Find all pure-strategy Nash equilibria.

`Equilibria are given by` $t_i = 0, t_j \in [\frac{1}{2}, 1]$

`Utility is given by`

$$u_i(t_i, t_j) = \begin{cases} -t_i & \text{if } t_i < t_j \\ -t_j + \frac{1}{2}(1 - t_j) & \text{if } t_i = t_j \\ -t_j + (1 - t_j) & \text{if } t_i > t_j \end{cases}$$

`which implies that driver` i `will select either` $t_i = 0$ `or` $t_i > t_j$. `Comparing profits of the two possibilities (`0 `and` $1-2t_j$`), we obtain best replies given by`

$$t_i(t_j) = \begin{cases} 0 & \text{if } t_j \geq \frac{1}{2} \\ (t_j, 1] & \text{if } t_j \leq \frac{1}{2} \end{cases}$$

with either being a best reply to $t_j = \frac{1}{2}$.