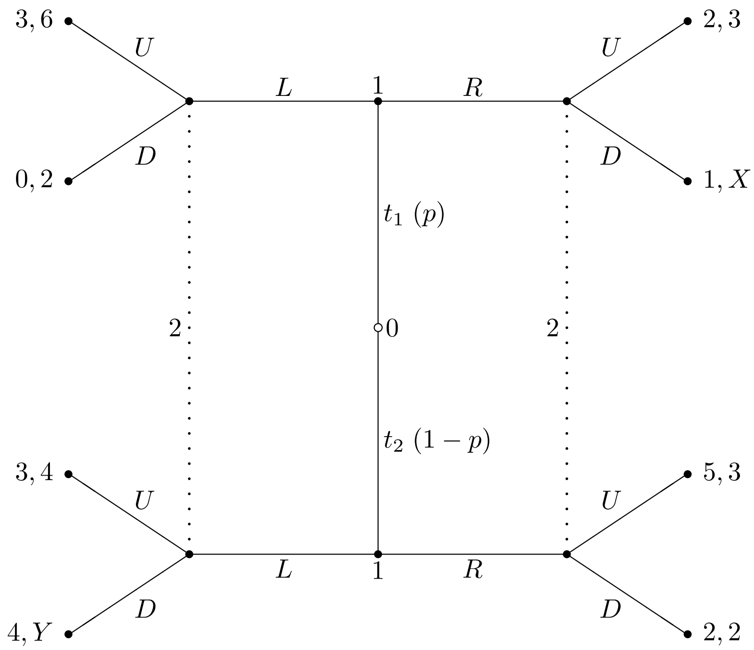
Microeconomic Theory II — Spring 2024

Final Exam Solutions — Mikhael Shor

Carefully explain and support your answers.

Question 1. Consider the following game. First, nature (player 0) selects t_1 with probability p, $0 < p < 1$, or t_2 with probability $1-p$. Next, player 1 selects L or R. Lastly, player 2 selects U or D.



(a) Assume $X = Y = 5$. Find all separating equilibria.

`There are two separating Equilibria:`

$\{t_1 \to L, t_2 \to R; L \to U, R \to U\}$ with beliefs $\mu(t_1|L) = \mu(t_2|R) = 1$.

`and`

$\{t_1 \to R, t_2 \to L; L \to D, R \to D\}$ with beliefs $\mu(t_1|R) = \mu(t_2|L) = 1$.

(b) Assume $p = \frac{1}{2}$. Find all values of X and Y for which there exists a pooling equilibrium on R. Carefully explain.

`Note first that we must have` $R \to U$ `or else` t_2 `will never play` R `and` $L \to D$ `or else` t_1 `will never play` R. `Therefore, a pooling equilibrium would require:`

$\{t_1 \to R, t_2 \to R; L \to D, R \to U\}$.

`We require beliefs for player 2 that support these strategies.`

`Following` R`, player 2's beliefs are equal to the initial probabilities` $(\frac{1}{2}, \frac{1}{2})$`. For` U `to be a best response:` $(\frac{1}{2})3 + (\frac{1}{2})3 \geq (\frac{1}{2})X + (\frac{1}{2})2$ `or` $X \leq 4$.

`Following` L`, beliefs are not constrained by Bayes Rule (as it is off the equilibrium path) so we can have` D `as a best response as long as there exists a` $\mu \in [0,1]$ `such that` $2\mu + Y(1-\mu) \geq 6\mu + 4(1-\mu)$ `or` $Y \geq \frac{4}{1-\mu}$`. This implies (letting` $\mu = 0$`) that` $Y \geq 4$`. There is any easier way to get this. For there to exist beliefs that make` D `a best response,` D `has to be a best response at (at least) one of the two decision nodes. This requires` $Y \geq 4$.

(c) Assume $p = \frac{1}{2}$. Find all values of X and Y for which there exists a pooling equilibrium on L. Carefully explain.

`Note first that we must have` $L \to U$ `or else` t_1 `will never play` L `and` $R \to D$ `or else` t_2 `will never play` L`. Therefore, a pooling equilibrium would require:`

$\{t_1 \to L, t_2 \to L; L \to U, R \to D\}$.

`We require beliefs for player 2 that support these strategies.`

`Following` L`, player 2's beliefs are equal to the initial probabilities` $(\frac{1}{2}, \frac{1}{2})$`. For` U `to be a best response:` $(\frac{1}{2})6 + (\frac{1}{2})4 \geq (\frac{1}{2})2 + (\frac{1}{2})Y$ `or` $Y \leq 8$.

`Following` R`, beliefs are not constrained by Bayes Rule (as it is off the equilibrium path) so we can have` D `as a best response as long as there exists a` $\mu \in [0,1]$ `such that` $X\mu + 2(1-\mu) \geq 3\mu + 3(1-\mu)$ `or` $X - 2 \geq \frac{1}{\mu}$`. This implies (letting` $\mu = 1$`) that` $X \geq 3$.

Note 1.1 The beliefs off the equilibrium path (following L in part b or following R in part c) are not constrained and, in particular, have nothing to do with p. There is no reason to assume that those beliefs are $\frac{1}{2}, \frac{1}{2}$.

Note 1.2 Some students left the condition for X or Y as a function of μ, the off-equilibrium beliefs. Note that, for an equilibrium to exist, you can choose whatever μ you want and thus should select the one that gives the greatest range for X or Y.

(d) Does the pooling equilibrium on L derived in the previous part satisfy the intuitive criterion? Carefully explain.

`1. There is an unsent message,` R.

`2. Type` t_1 `would never send` R `as the equilibrium payoffs, 3 are strictly higher than any payoffs obtainable from` R`, 2 or 1.`

3. Therefore, we require $\mu(t_1|R) = 0$. This makes U the best response to R. Lastly, type t_2 is strictly better off from R (payoff of 5) than in equilibrium (3).

All conditions for renegotiation are satisfied, so this equilibrium does not satisfy the intuitive criterion.

Question 2. Consider a principal-agent problem in which the agent chooses between two levels of effort, $\{e_l, e_h\}$. The principal pays the agent a wage $w_s \geq 0$ in state s and realizes output of π_s. There are four states, with $(\pi_1, \pi_2, \pi_3, \pi_4) = (500, 430, 20, 0)$. The probability of a state s (or output π_s) contingent on effort is given by:

		output			
		π_1	π_2	π_3	π_4
effort level	e_h	.3	.3	.3	.1
	e_l	.1	.1	.6	.2

The agent's utility function is $\sqrt{w} - c(e)$ where $c(e_h) = 4, c(e_l) = 0$, and the agent's reservation utility is $\underline{u} = 1$. The principal is risk neutral, with utility given by $\pi - w$.
Wages may not be negative.

(a) Determine the wage schedule that optimally implements e_h when effort is observable.

When effort is observable, we have only the IR constraint: $\sqrt{w} \geq c(e_l) + \underline{u} = 5$ which binds at the optimal wage, so we want $w_s = 25\ \forall s$ when effort is high and (say) $w_s = 0$ when effort is low.

(b) Determine the wage schedule that optimally implements e_h when effort is unobservable.

First, we note that $w_1 = w_2$ and $w_3 = w_4$ because the ratios of probabilities under low and high effort are the same. Then, we have four constraints:

IC: $.6\sqrt{w_1} + .4\sqrt{w_3} - 4 \geq .2\sqrt{w_1} + .8\sqrt{w_3}$ or equivalently, $\sqrt{w_1} \geq \sqrt{w_3} + 10$

IR0: $.6\sqrt{w_1} + .4\sqrt{w_3} - 4 \geq 1$

IR1: $w_1 \geq 0$

IR2: $w_3 \geq 0$

Note that IC and IR2 imply that $w_1 \geq 100$ and thus imply both IR1 and IR0. Therefore, both IC and IR2 must bind. This yields $w_3 = w_4 = 0, w_1 = w_2 = 100$.

(c) When effort is unobservable, what effort level does the principal wish to implement? Explain.

First note that the optimal wage for low effort is simply a constant wage satisfying $\sqrt{w} - c(e_l) = \underline{u}$ or $w = 1$.

Then, profit for high effort is:

$$.3(500)+.3(430)+.3(20)+.1(0)-.3(100)-.3(100)-.3(0)-.1(0) = 285-60 = 225.$$

and profit for low effort is:

$$.1(500) + .1(430) + .6(20) + .2(0) - 1 = 105 - 1 = 104$$

Therefore, inducing high effort is optimal, so the optimal wage is $w_3 = w_4 = 0, w_1 = w_2 = 100$.

(d) Imagine that the government institutes a minimum wage, $\hat{w}$, requiring that $w(\pi) \geq \hat{w} \ \forall \pi$. Assume that effort is unobservable. Show that the principal is indifferent between implementing high effort and low effort when $\hat{w} = 100$.

For low effort, the minimum wage will bind so $w_s = 100$ in every state. Profit is given by:

$$.1(500) + .1(430) + .6(20) + .2(0) - 100 = 105 - 100 = 5$$

For high effort, minimum wage replaces our previously binding IR constraint, so our two binding constraints are:

$$IR : w_3 \geq 100$$

and

$$IC : \sqrt{w_1} \geq \sqrt{w_3} + 10$$

With $w_1 = w_2$ and $w_3 = w_4$ as before. This yields $w_1 = w_2 = 400, w_3 = w_4 = 100$. Profit is given by:

$$.3(500) + .3(430) + .3(20) + .1(0) - .3(400) - .3(400) - .3(100) - .1(100) = 285 - 280 = 5.$$

Question 3. Two identical firms (1 and 2) produce a homogeneous product. Competition takes place over two periods. In the first period, each firm simultaneously selects a level of advertisement, $a_i \geq 0, i \in \{1, 2\}$, with costs given by $c(a_i) = \frac{1}{3}a_i^2$. In the second period, after observing first-period advertising expenditures, each firm selects a quantity, q_i. The industry inverse demand function is given by $p = a_1 + a_2 - q_1 - q_2$. Marginal costs of production are zero. Each firm maximizes profit, given by $pq_i - c(a_i)$.

(a) What is the subgame perfect equilibrium? Carefully show all work.

Firm 1's profit is given by $(a_1 + a_2 - q_1 - q_2)q_1 - \frac{1}{3}a_1^2$

We begin with the second stage:

$$\frac{d\pi_1}{dq_1} = (a_1 + a_2 - 2q_1 - q_2) = 0$$

which yields the best response function:

$$q_1(q_2; a_1, a_2) = \frac{a_1 + a_2}{2} - \frac{1}{2}q_2$$

and by symmetry

$$q_2(q_1; a_1, a_2) = \frac{a_1 + a_2}{2} - \frac{1}{2}q_1$$

and the familiar Cournot solution:

$$q_1^*(a_1, a_2) = q_2^*(a_1, a_2) = \frac{1}{3}(a_1 + a_2)$$

Next, we consider the first stage. Substituting the second stage strategies into firm 1's profit function yields

$$(a_1 + a_2 - q_1 - q_2)q_1 - \frac{1}{3}a_1^2 = \frac{1}{9}(a_1 + a_2)^2 - \frac{1}{3}a_1^2$$

Maximizing,

$$\frac{d\pi_1}{da_1} = \frac{2}{9}(a_1 + a_2) - \frac{2}{3}a_1 = 0$$

which yields the best response function:

$$a_1(a_2) = \frac{1}{2}a_2 \text{ and by symmetry } a_2(a_1) = \frac{1}{2}a_1$$

and a solution of $a_1 = a_2 = 0$.

Therefore, the SPNE is:

$$\left\{a_1 = 0, q_1(a_1, a_2) = \frac{1}{3}(a_1 + a_2); a_2 = 0, q_2(a_1, a_2) = \frac{1}{3}(a_1 + a_2)\right\}$$

(b) What are each firm's equilibrium profits?

With $a_1 = a_2 = 0$, profits are 0

Note 3.1 Some students solved only the second period in part (a) and continued to solve the first period in part (b). Part (a) asked for the subgame-perfect Nash equilibrium, which requires solving both periods.

Note 3.2 A strategy for a player consists of a decision in each period: (i) a numerical solution for a_i and (ii) a solution for q_i for every subgame (for every value of a_1 and a_2.)

(c) Imagine that each firm set its level of advertising to 1 ($a_1 = a_2 = 1$) in the first period, and then played its Nash equilibrium strategy in the second period. What would be each firm's profit?

We know from above that $q_i = \frac{1}{3}(a_1 + a_2)$ so $q_1 = q_2 = \frac{2}{3}$. Substituting into profit:

$$(a_1 + a_2 - q_1 - q_2)q_1 - \frac{1}{3}a_1^2 = \left(\frac{2}{3}\right)^2 - \frac{1}{3} = \frac{1}{9}$$

(d) Does there exist a Nash equilibrium in which both firms set their levels of advertising to 1? Briefly explain why or why not.

`This question was looking for awareness of the difference between SPNE and NE. NE allows for non-optimal actions off the equilibrium path. For example, consider a strategy of` $a_i = 1$, $q_i = \frac{2}{3}$ `when` $a_1 = a_2 = 1$ `and` $q_i = 100$ `for all other` a_i. `In response for firm` j, `this implies any` a_j `other than` 1 `yields profits of 0 (as the other firm's quantity is too large or sufficiently large` a_j `is too expensive).`